Confined Maxwell Field and Temperature Inversion Symmetry

F. C. Santos* and A. C. Tort†
Instituto de Física
Universidade Federal do Rio de Janeiro

Cidade Universitária, C.P. 68528

CEP 21945-970 Rio de Janeiro RJ, Brazil

February 1, 2008

Abstract

We evaluate the Casimir vacuum energy at finite temperature associated with the Maxwell field confined by a perfectly conducting rectangular cavity and show that an extended version of the temperature inversion symmetry is present in this system.

PACS: 11. 10. -z; 11. 10. Wx

*filadelf@if.ufrj.br
†tort@if.ufrj.br

1 Introduction.

Temperature inversion symmetry occurs in the free energy associated with the Casimir effect [1] at finite temperature and it is linked to the nature of the boundary conditions imposed on the quantum vacuum oscillations. As shown by Ravndal and Tollefsen [2], temperature inversion symmetry holds for massless bosonic fields and symmetric boundary conditions and also for massless fermionic fields and antisymmetric boundary conditions. Temperature inversion symmetry was initially studied by Brown and Maclay [3] who wrote the scaled free energy associated with the Casimir effect at finite temperature as a sum of three contributions, namely: a zero temperature contribution, that is, the Casimir energy density at zero temperature, a Stefan-Boltzmann contribution proportional the fourth power of the scaled temperature $\xi := Td/\pi$, and a non-trivial contribution. The three contributions to the scaled free energy can be combined into a single double sum:

$$f(\xi) = -\frac{1}{16\pi^2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{(2\pi\xi)^4}{[m^2 + (2\pi n\xi)^2]^2}\,, \tag{1}$$

where the term corresponding to $n = m = 0$ is excluded. Setting $n = 0$ and summing over m with $m \neq 0$, we obtain the Stefan-Boltzmann limit $-\pi^6\xi^4/45$. On the other hand, setting $m = 0$ and summing over n with $n \neq 0$, we obtain the zero temperature Casimir term $-\pi^2/720$. This function has the following property:

$$(2\pi\xi)^4 f\left(1/4\pi\xi\right) = f\left(\xi\right), \tag{2}$$

which is the mathematical statement of the temperature inversion symmetry. It was also shown by Gundersen and Ravndal [4] that the scaled free energy associated with massless fermions fields at finite temperature submitted to MIT boundary conditions satisfy the relation given by Eq.(2) and therefore exhibts temperature inversion symmetry. Tadaki and Takagi [5] have calculated Casimir free energies for a massless scalar field obeying Dirichlet or Neumann boundary conditions on both plates and found this symmetry. For the parallel plate geometry and mixed boundary conditions it is possible to circunvent the restrictions found by [2] and discuss temperature inversion symmetry by reconizing that with respect to the evaluation of the free energy this arrangement is equivalent to the difference between two Dirichlet (or Neumann) plates [6]. In the case of a massless scalar field at finite temperature and periodic boundary conditions, it is possible to show that the partition function, and consequently the free energy, can be written in a closed form such that the temperature inversion symmetry becomes explicit [7]. Here we shall show that for the case of the Maxwell field confined by a perfectly conduting rectangular cavity this symmetry is also present, provided that we generalize the transformations that relate the high and the low temperature regimes.

Throughout this letter we employ units such that Boltzmann constant k_B, the speed of light c and $\hbar = h/2\pi$ are set equal to the unity.

2 Evaluation of Helmholtz free energy

Helmholtz free energy for the confined Maxwell field within a perfectly conducting rectangular box can be evaluated by means of the generalized zeta function technique [9], [10]. First we introduce the global zeta function which is defined by

$$\mu^{2s}\zeta(s) = \mu^{2s} \sum_{p=-\infty}^{\infty} \sum_{\{n\}} \left[\left(\frac{2\pi p}{\beta} \right)^2 + \omega_{\{n\}}^2 \right]^{-s} \tag{3}$$

where $\beta = 1/T$, the reciprocal of the temperature T, is the periodic length along the Euclidean time direction and a mass scale parameter μ was introduced in order to keep Eq.(3) dimensionless; $\omega_{\{n\}}^2$ are the eigenvalues associated with the Euclidean time-independent modal equation

$$-\triangle_{\mathbf{x}}\, \varphi_{\{n\}}(\mathbf{x}) = \omega_{\{n\}}^2\, \varphi_{\{n\}}(\mathbf{x}), \tag{4}$$

where $\mathbf{x} = (x, y, z)$ and $\triangle_{\mathbf{x}}$ is the Laplacian operator. The Helmholtz free energy function $F(\beta)$ for bosonic fields can be obtained from the knowledge of the global zeta function through the relation

$$F(\beta) = -\frac{1}{2\beta} \frac{\partial}{\partial s} \left[\mu^{2s}\zeta(s) \right]_{s=0}. \tag{5}$$

The eigenfrequencies associated with the allowed electromagnetic normal modes within a perfectly conducting rectangular box of linear dimensions a, b, c are given by

$$\omega_{lmn}^2 = \left(\frac{l\pi}{a} \right)^2 + \left(\frac{m\pi}{b} \right)^2 + \left(\frac{n\pi}{c} \right)^2, \tag{6}$$

where $l, m, n \in \{1, 2, \ldots\}$. For $l, n, m \neq 0$ there are two possible polarization states and for $l = 0$ or $m = 0$ or $n = 0$ there is one polarization state only. Eigenfrequencies for which three or two of the integers l, m, n are simultaneously zero are not allowed. It follows that the generalized zeta function for the case in question reads

$$\begin{aligned}
\mu^{2s}\zeta(s) &= 2\mu^{2s} \sum_{l,m,n=1}^{\infty} \sum_{p=-\infty}^{\infty} \left[\left(\frac{l\pi}{a} \right)^2 + \left(\frac{m\pi}{b} \right)^2 + \left(\frac{n\pi}{c} \right)^2 + \left(\frac{2p\pi}{\beta} \right)^2 \right]^{-s} \\
&\quad + \mu^{2s} \sum_{l,m=1}^{\infty} \sum_{p=-\infty}^{\infty} \left[\left(\frac{l\pi}{a} \right)^2 + \left(\frac{m\pi}{b} \right)^2 + \left(\frac{2p\pi}{\beta} \right)^2 \right]^{-s}
\end{aligned}$$

$$+\mu^{2s}\sum_{l,n=1}^{\infty}\sum_{p=-\infty}^{\infty}\left[\left(\frac{l\pi}{a}\right)^2+\left(\frac{n\pi}{c}\right)^2+\left(\frac{2p\pi}{\beta}\right)^2\right]^{-s}$$

$$+\mu^{2s}\sum_{m,n=1}^{\infty}\sum_{p=-\infty}^{\infty}\left[\left(\frac{m\pi}{b}\right)^2+\left(\frac{n\pi}{c}\right)^2+\left(\frac{2p\pi}{\beta}\right)^2\right]^{-s}, \qquad (7)$$

If we now separate the terms corresponding to the zero temperature sector by setting $p=0$, then it will be easily seen that we can rewrite Eq.(7) formally as a sum of Epstein functions (see the Appendix) which reads

$$\begin{aligned} 2^2\left(\frac{\pi}{\mu}\right)^{2s}\zeta(s) &= 2^4E_4\left(s;1/a^2,1/b^2,1/c^2,4/\beta^2\right)+2^3E_3\left(s;1/a^2,1/b^2,1/c^2\right) \\ &\quad +2^3E_3\left(s;1/a^2,1/b^2,4/\beta^2\right)+2^2E_2\left(s;1/a^2,1/b^2\right) \\ &\quad +2^3E_3\left(s;1/a^2,1/c^2,4/\beta^2\right)+2^2E_2\left(s;1/a^2,1/c^2\right) \\ &\quad +2^3E_3\left(s;1/b^2,1/c^2,4/\beta^2\right)+2^2E_2\left(s;1/b^2,1/c^2\right) \end{aligned} \qquad (8)$$

In order to regularize Eq.(8) it is convenient to rewrite it as a sum of generalized zeta functions and this can be accomplished by making use of Eqs. (4) and (26) in the appendix. Equation (8) then takes the form

$$\begin{aligned} 2^2\left(\frac{\pi}{\mu}\right)^{2s}\zeta(s) &= A_4\left(s;1/a^2,1/b^2,1/c^2,4/\beta^2\right)-A_2\left(s;1/a^2,4/\beta^2\right) \\ &\quad -A_2\left(s;1/b^2,4/\beta^2\right)-A_2\left(s;1/c^2,4/\beta^2\right)+4E_1\left(s;4/\beta^2\right). \end{aligned} \qquad (9)$$

Finally, we can regularize Eq.(9) by applying the reflection formula, Eq.(4) on all terms except the last one which is already regularized thus obtaining

$$\begin{aligned} \mu^{2s}\zeta(s) &= \frac{\mu^{2s}abc\beta}{8\pi^2}\frac{\Gamma(2-s)}{\Gamma(s)}A_4\left(2-s;a^2,b^2,c^2,\beta^2/4\right) \\ &\quad -\frac{\mu^{2s}a\beta}{8\pi^2}\frac{\Gamma(1-s)}{\Gamma(s)}A_2\left(1-s;a^2,\beta^2/4\right) \\ &\quad -\frac{\mu^{2s}b\beta}{8\pi^2}\frac{\Gamma(1-s)}{\Gamma(s)}A_2\left(1-s;b^2,\beta^2/4\right) \\ &\quad -\frac{\mu^{2s}c\beta}{8\pi^2}\frac{\Gamma(1-s)}{\Gamma(s)}A_2\left(1-s;c^2,\beta^2/4\right)+\left(\frac{\beta}{2}\right)^{2s}\zeta_R(2z), \end{aligned} \qquad (10)$$

where $\zeta_R(2z)$ is the Riemann zeta function and we have also made use of the fact that $E_1(z;a) = a^{-z}\zeta_R(2z)$. Notice that all terms on the R.H.S. of Eq.(10) are zero when

$s \rightarrow 0$ except the last one, that is, $\zeta(0) \neq 0$ since $\zeta_R(0) = -(1/2)\ln(2\pi)$. This will give rise to a scale-dependent term absent in previous calculations related to this problem [15]. By making use of Eq.(5) we see that the Helmholtz free energy for the confined electromagnetic field confined within a perfectly conducting rectangular box at finite temperature is

$$
\begin{aligned}
F\,(a,b,c,\beta) \;=\; & -\frac{abc}{16\pi^2} \sum_{l,m,n,p=-\infty}^{\infty} \frac{1}{\left(a^2l^2 + b^2m^2 + c^2n^2 + \frac{p^2\beta^2}{4}\right)^2} \\
& + \frac{a}{16\pi} \sum_{l,p=-\infty}^{\infty} \frac{1}{\left(a^2l^2 + \frac{p^2\beta^2}{4}\right)} + \frac{b}{16\pi} \sum_{m,p=-\infty}^{\infty} \frac{1}{\left(b^2m^2 + \frac{p^2\beta^2}{4}\right)} \\
& + \frac{c}{16\pi} \sum_{n,p=-\infty}^{\infty} \frac{1}{\left(c^2n^2 + \frac{p^2\beta^2}{4}\right)} + \frac{1}{\beta} \ln\left(\frac{\mu\beta}{\sqrt{2\pi}}\right). \qquad (11)
\end{aligned}
$$

The presence of a scale-dependent term is a leftover from the renormalization procedure carried on via the generalized zeta method and introduces an ambiguity in the Helmholtz free energy related to the problem in question. This ambiguity is introduced whenever $\zeta(s)$ is not zero at $s = 0$, see for instance[10]. It was shown by Myers [11] that this ambiguty is a legitimate feature of the generalized zeta function regularization method, whenever the theory is massive or/and interacting and/or some dimensions are compactified. As consequence the usual mode sum method and the generalized zeta function method are not always equivalent. However, the usual mode sum method can be modified so as to include the complexities of a scale dependence [12]. At zero temperature and for the simple geometrical cavity that we are considering here scale-dependent terms are not expected. This is confirmed by the independent calculations due to Lukosz [13], Ruggiero *et al* [14], Ambjorn and Wolfram [15] and the present authors. Here it is clear that the scale dependence is introduced by the generalized zeta function regularization method due to the imposition of periodic conditions on the Euclidean time direction. For our purposes the ambiguity in the free energy can be solved by the additional physical requirement that in unconstrained space, that is, for a very large rectangular box, the only surviving term must be the Stefan-Boltzmann term. It is seen then that μ should be equal to $\sqrt{2\pi}/\beta$.

3 Temperature inversion symmetry

The explicit verification is of temperature inversion symmetry in the case of the system in question is complicated by the fact that in addition to the inverse temperature parameter β we have to deal with three other length parameters, namely, a, b and c, the measures of the sides of the rectangular cavity. At finite temperature and periodic or antiperiodic conditions along one spatial direction, or Dirichlet, or Neumann plane surfaces located

perpendicularly to one spatial direction only two characteristic lengths are involved, a feature that rends this verification easier. Neverheless, by making recourse to a simple trick we shall show that temperature inversion symmetry is also present here.

First notice that the denominator of the first term on the R.H.S of Eq.(11) can be rewritten as $a^2l^2 + b^2m^2 + c^2n^2 = \kappa^2\left(a^2q^2 + b^2r^2 + c^2t^2\right)$, where $\{q, r, t\}$ is a sequence of three integers with no common factor and κ is the common factor of $\{l, m.n\}$. For the sequence $\{q, r, t\}$ we define a characteristic length $d_{\{q,r,t\}}$ by

$$d^2_{\{q,r,t\}} := a^2q^2 + b^2r^2 + c^2t^2. \tag{12}$$

Let us also define the dimensionless variable

$$\xi_{\{q,r,t\}} := \frac{2d_{\{q,r,t\}}}{\beta}. \tag{13}$$

The remaining terms on the R.H.S. of Eq.(11) can be treated in a simpler way. For example, for the second term we define

$$\xi_a := \frac{2a}{\beta}, \tag{14}$$

Then making the replacement: $\sum_{l,m,n=-\infty} \to \sum_{\{q,r,t\}}\sum_{\kappa=-\infty}^{+\infty}$, the free energy density corresponding to Eq.(11) can be written as

$$\begin{aligned}\frac{F\,(a,b,c,\beta)}{abc} \quad = \quad & -\frac{1}{16\pi^2}\sum_{\{q,r,t\}}\sum_{\kappa,\,p=-\infty}^{\infty}\frac{1}{d^4_{\{q,r,t\}}}\frac{\xi^4_{\{q,r,t\}}}{\left(\kappa^2\xi^2_{\{q,r,t\}}+p^2\right)^2}\\ & +\frac{1}{16\pi a^2\,bc}\sum_{l,p=-\infty}^{\infty}\frac{\xi_a^2}{(l_a^2\xi_a^2+p^2)}\\ & +\frac{1}{16\pi a\,b^2c}\sum_{m,p=-\infty}^{\infty}\frac{\xi_b^2}{(m_b^2\xi_b^2+p^2)}\\ & +\frac{1}{16\pi abc^2}\sum_{n,p=-\infty}^{\infty}\frac{\xi_c^2}{(n^2\xi_c^2+p^2)}\end{aligned} \tag{15}$$

The zero temperature limit of Eq.(15) is obtained by reconizing that when $\beta \to \infty$ the dimensionless parameters $\xi_{\{q,r,t\}}, \xi_a, \xi_b, \xi_c \to 0$ and the surviving terms in Eq.(15) correspond to $p = 0$, as the reader can easily verify. Going back to our initial set of indexes l, m, n we obtain

$$E_0 == -\frac{abc}{16\pi^2}\sum_{l,m,n=-\infty}^{+\infty}\frac{1}{\left[a^2l^2+b^2m^2+c^2n^2\right]^2}+\frac{\pi}{48}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right), \tag{16}$$

which is in perfect agreement with the result obtained by Lukosz [13] and also by Ruggiero *et al.*[14].

In order to introduce temperature inversion symmetry for this problem let us now define the dimensionless functions

$$\mathcal{F}_{\{q,r,t\}}\left(\xi_{\{q,r,t\}}\right) := -\frac{1}{16\pi^2}\sum_{\kappa,\,p=-\infty}^{\infty}\frac{\mathcal{T}^4_{\{q,r,t\}}}{\left(\kappa^2\xi^2_{\{q,r,t\}}+p^2\right)^2}, \tag{17}$$

and

$$\mathcal{F}_a\left(\xi_a\right) := \frac{1}{16\pi}\sum_{l,p=-\infty}^{\infty}\frac{\xi_a^2}{\left(l_a^2\xi_a^2+p^2\right)}, \tag{18}$$

with similar definitions for $\mathcal{F}_b\left(\xi_b\right)$ and $\mathcal{F}_c\left(\xi_c\right)$. Therefore, in terms of these functions we can write the free energy density as

$$\frac{F}{V} = \sum_{\{q,r,t\}}\frac{\mathcal{F}_{\{q,r,t\}}\left(\xi_{\{q,r,t\}}\right)}{d^4_{\{q,r,t\}}} + \frac{\mathcal{F}_a\left(\xi_a\right)}{aV} + \frac{\mathcal{F}_b\left(\mathcal{T}_b\right)}{bV} + \frac{\mathcal{F}_c\left(\xi_c\right)}{cV}, \tag{19}$$

where $V = abc$. It is not hard to see that the functions $\mathcal{F}_{\{q,r,t\}}\left(\xi_{\{q,r,t\}}\right)$ exhibit the following property

$$\xi^4_{\{q,r,t\}}\mathcal{F}_{\{q,r,t\}}\left(\frac{1}{\xi_{\{q,r,t\}}}\right) = \mathcal{F}_{\{q,r,t\}}\left(\xi_{\{q,r,t\}}\right). \tag{20}$$

In the same way

$$\xi_a^2\ f_a\left(\frac{1}{\xi_a}\right) = f_a\left(\xi_a\right), \tag{21}$$

Equations (20) and (21) plus two similar equations for the functions $\mathcal{F}_b\left(\xi_b\right)$ and $\mathcal{F}_c\left(\xi_c\right)$ describe temperature inversion symmetry for the case in question, that is, all terms in Eq.(19) can be inverted by making use of these formulae.

In the very high temperature limit we expect the leading contribution to be the Stefan-Boltzmann term, $\pi^2/\left(45\beta^4\right)$. Now we argue that our transformations are applied in order to generate from the unique Stefan-Boltzmann density term, an infinite number of terms which must be added at zero temperature, and inversely, each term at zero temperature goes to the unique Stefan-Boltzmann term. In this form, that is, as far as the energy density is concerned, we can apply the temperature inversion symmetry transformations to several Casimir systems, including of course the ones previously analyzed in the literature. The specific form of the transformations depends on the particular system at hand, but the procedure is the same.

Now, we can easily check that $\mathcal{F}_{\{q,r,t\}}(0) = -\pi^2/720$, and that $\mathcal{F}_a(\xi_a) = \mathcal{F}_b(\xi_b) = \mathcal{F}_c(\xi_c) = \pi/48$. Then, making use of Eqs.(20) and (21) we write

$$\xi^4_{\{q,r,t\}} \mathcal{F}_{\{q,r,t\}}(\infty) = -\frac{\pi^2}{720}, \tag{22}$$

and

$$\xi_a^2\, \mathcal{F}_a(\infty) = \xi_b^2\, \mathcal{F}_b(\infty) = \xi_c^2\, \mathcal{F}_c(\infty) = \frac{\pi}{48}. \tag{23}$$

Taking these results into Eq.(19) we obtain

$$F(a,b,c,\beta \to 0) \approx -\frac{abc\,\pi^2}{45\beta^4} + \frac{\pi}{12\beta^2}(a+b+c), \tag{24}$$

which is in agreement with [15] with respect to the leading terms in the high temperature approximation. As expected, the Stefan-Boltzmann term is the leading term in this limit.

It is convenient to rewrite Eq.(5), or its equivalent Eq.(19), with the zero temperature terms and the Stefan-Boltzmann term separated from the non-trivial temperature corrections,

$$\begin{aligned}
\frac{F}{V} \;=\;& -\frac{\pi^2}{45\beta^4} + \frac{\pi}{12\beta^2}\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) \\
& -\frac{1}{16\pi^2} \sum_{l,m,n=-\infty}^{+\infty} \frac{1}{\left[a^2l^2 + b^2m^2 + c^2n^2\right]^2} + \frac{\pi}{48abc}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \\
& -\frac{1}{4\pi^2} \sum_{\{q,r,t\}} \sum_{\kappa,p=1}^{\infty} \frac{1}{\left(\kappa^2 d^2_{\{q,r,t\}} + \beta^2p^2\right)^2} + \frac{1}{\pi bc} \sum_{l,p=1}^{\infty} \frac{1}{(4a^2l^2 + \beta^2p^2)} \\
& + \frac{1}{\pi ac} \sum_{l,p=1}^{\infty} \frac{1}{(4b^2l^2 + \beta^2p^2)} + \frac{1}{\pi ab} \sum_{l,p=1}^{\infty} \frac{1}{(4c^2l^2 + \beta^2p^2)}.
\end{aligned} \tag{25}$$

The last two lines of Eq.(25) represent the non-trivial temperature-dependent corrections. The first term in the third line is equivalent to a set of conducting plates with each pair of plates characterized by a distance $d_{\{q,r,t\}}$, which means that we can take advantage of results already established in the literature regarding high and low temperature approximations for a pair of conducting plates.

4 Conclusions

In this letter we have employed generalized zeta function techniques in its global version to derive the finite temperature vacuum energy associated with an electromagnetic field

confined within a perfectly conducting rectangular box. We have shown that Helmholtz free energy for the problem in question is scale dependent, a feature that was not present in previous calculations concerning this problem at zero temperature [13], [14] and finite temperature [15]. The scale dependence is a remainder that tell us that the theory was renormalized by the way of the generalized zeta function method. We have also shown that the concept of temperature inversion symmetry can be extended so as to include the elctromagnetic field confined by perfectly conducting rectangular cavity. The main point here is the recognition that the concept of temperature inversion symmetry can be extended in the sense that it must be applied separately to the different terms comprising the non-trivial part of the free energy density, with each term behaving differently under this symmetry. This is the content of Eqs. (20) and (21). The most important feature of temperature inversion symmetry is that it allows us to establish a relationship between the zero and low-temperature sector of the Helmholtz free energy and the high-temperature one. This can be very useful since in general it is easier to obtain low-temperature expansions. In particular, it is possible to relate the zero temperature Casimir effect to the Stefan-Boltzmann term in a straightforward way.

Appendix

Multidimensional homogeneous Epstein function [8] (see also Elizalde *et al.*[10] for a modern presentation) are defined by

$$E_N\left(z; a_1, a_2, ...a_N\right) = \sum_{n_1, n_2, ..n_N=1}^{\infty} \left(a_1 n_1^2 + a_2 n_2^2 + \cdots + a_N\, n_N^2\right)^{-z}, \qquad (A.1)$$

for $\Re z > N/2$ and $a_1, a_2, ...a_N > 0$. Generalized zeta functions. $A_N\left(z; a_1, a_2, ...a_N\right)$ are defined by [8],[10]

$$A_N\ \left(z; a_1, a_2, ...a_N\right) = \sum_{n_1, n_2, ..n_N=-\infty}^{+\infty}{}' \left(a_1 n_1^2 + a_2\, n_2^2 + \cdots + a_N\, n_N^2\right)^{-z}, \qquad (A.2)$$

for $\Re z > N/2$ and $a_1, a_2, ...a_N > 0$. The prime means that the term corresponding to $n_1 = n_2 = ... = n_N = 0$ must be excluded from the summation. A useful reflection formula reads envolving these functions is [10]

$$A_N\ \left(z; a_1, a_2, ...a_N\right) = \frac{\pi^{-\frac{N}{2}+2z}}{\sqrt{a_1...a_N}} \frac{\Gamma\left(\frac{N}{2} - z\right)}{\Gamma\left(z\right)} A_N\ \left(\frac{N}{2} - z; 1/a_1, 1/a_2, ..1/.a_N\right). \qquad (A.3)$$

Relationships between generalized zeta functions and Epstein functions can be derived from the very definition given by (4). For example,

$$A_2\left(z; a_1, a_2\right) = 2^2 E_2\left(z; a_1, a_2\right) + 2E_1\left(z; a_1\right) + 2E_1\left(z; a_2\right) \qquad (A.4)$$

and

$$
\begin{aligned}
&A_4\left(z;a_1,a_2,a_3,a_4\right)=2^4E_4\left(z;a_1,a_2,a_3,a_4\right)+2^3E_3\left(z;a_1,a_2,a_3\right)\\
&+2^3E_3\left(z;a_1,a_2,a_4\right)+2^3E_3\left(z;a_1,a_3,a_4\right)+2^2E_2\left(z;a_1,a_2\right)+2^2E_2\left(z;a_1,a_3\right)\\
&+2^3E_3\left(z;a_2,a_3,a_4\right)+2^2E_2\left(z;a_2,a_3\right)+2^2E_2\left(z;a_1,a_4\right)+2^2E_2\left(z;a_2,a_4\right)\\
&+2^2E_2\left(z;a_3,a_4\right)+2E_1\left(z;a_1\right)+2E1\left(z;a_2\right)+2E1\left(z;a_3\right)+2E_1\left(z;a_4\right), \qquad (A.5)
\end{aligned}
$$

of which use was made in this letter.

Acknowledgments

The authors wish to acknowledge useful converstions with Dr. A. A. Actor.

References

[1] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. 51, 793 (1948); Philips Res.Rep. 6. 162 (1951).

[2] F. Ravndal and D Tollefsen, Phys. Rev. D **40**, 4191 (1989).

[3] L. S. Brown and G. J. Maclay Phys. Rev. **184** 1272 (1969).

[4] S. A. Gundersen and F. Ravndal, Ann. of Phys. **182**, 90 (1988).

[5] S. Tadaki and S. Takagi, Progr. Theor. Phys. **75** 262 (1986).

[6] F.C. Santos, A. Tenório and A.C. Tort, Phys. Rev. D **60** 105022 (1999).

[7] C. Wotzasek, J. Phys. A **23**, 1627 (1990).

[8] P. Epstein, Math. Ann. 56, 615 (1902); ; Math. Ann. **63**, 205 (1907).

[9] A. Salam and J. Strathdee, Nuc. Phys B **90**, 203 (1975); J. S. Dowker and R. Critchley, Phys. Rev. D **13**, 3224 (1976); S. W. Hawking, Commun. Math. Phys. **55**, 133 (1977); G. W. Gibbons, Phys. Lett. A **60**, 385 (1977).

[10] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko and S. Zerbini: *Zeta Regularization Techniques with Applications*, World Scientific, Singapore (1994).

[11] E. Myers, Phys. Rev. Lett. **59**, 165 (1987).

[12] S.K. Blau, M. Visser and A. Wipf, Nucl. Phys. **B310**, 163 (1988); See also L. C. Albuquerque, *Renormalization Ambiguities in Casimir Energy*, hep-th/9803223.

[13] W. Lukosz, Physica **56**, 109 (1971).

[14] R. Ruggiero, A. H. Zimerman and A. Villani, Rev. Bras. Fís. **7**, 663 (1977).

[15] J. Ambjorn and S. Wolfram, Ann. of Phys. **147**, 1 (1983).